Fundamental of ODEs

- Last time:

- $f: \mathbb{R}^n \to \mathbb{R}^n$ is locally Lip. if for all compact subsets $D \subset \mathbb{R}^n$, $\exists\, L > 0$ s.t.

$$\|f(x) - f(y)\| \leq L\|x - y\| \quad \forall x, y \in D$$

- It is globally Lip. if the inequality is true everywhere

$$\|f(x) - f(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathbb{R}^n$$

- If $f(t, x)$ is also a function of time, then $f: [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lip in x uniformly on $t \in [t_0, t_1]$ if $\forall$ compact subsets $D \subset \mathbb{R}^n$

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad \forall x, y \in D, \ \forall t \in [t_0, t_1]$$

- globally Lip $\rightsquigarrow$ $\|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [t_0, t_1]$

Thm : (Existence and uniqueness)

Consider ODE $\quad \dot{x} = f(t, x), \quad x(t_0) = x_0$

ordinary differential equation $\qquad (*)$

Assume

- f is piecewise continuous in t.
- locally Lip. in x uniformly in $t \in [t_0, t_1]$

would depend on x_0

Then, for all $x_0 \in \mathbb{R}^n$, $\exists \delta > 0$ s.t.
there exists a unique solution to $(*)$
on the interval $[t_0, t_0 + \delta]$

- If $f(t,x)$ is globally Lip. unif in $t \in [t_0, \infty)$
Then, the solution exists for all time.

Remark : [Khalil footnote 3 pp 88]

- If we only care about existence, not uniqueness, then continuity of $f(x)$ is enough.

e.g. $f(x) = \sqrt{x}$

proof:

Contraction mapping theorem

- we will use CMT to prove existence and uniqueness.

- we have to write the equation as a fixed point problem.

- Integrate the ODE to get

$$X(t) = X_0 + \int_{t_0}^{t} f(s, X(s))\, ds \qquad \forall t \in [t_0, t_1]$$

- Consider the space $\mathcal{X} = C([t_0, t_1]; \mathbb{R}^n)$ with $\|\cdot\|_\infty$-norm

- Consider the map

$$P : \mathcal{X} \to \mathcal{X}$$

it maps a continuous funct. to a continuous funct.

$$(PX)(t) \triangleq X_0 + \int_{t_0}^{t} f(s, X(s))\, ds$$

$$\forall t \in [t_0, t_1]$$

- The solution to the ODE is a fixed point of

$$X = PX$$

- we will use CMT to prove existence and uniqueness of solution.

- Conditions for CMT

(I) Define a closed subset $S \subseteq \mathcal{X}$

(II) $P(X) \in S$ for all $X \in S$

(III) $\|P(X) - P(Y)\| \leq \rho \|X - Y\|$

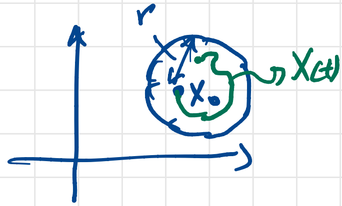
where $\rho \in (0, 1)$

- start with (I)

to be determined later

$$S = \{ X \in C([t_0, t_0 + \delta]; \mathbb{R}^n) \mid \|X - X_0\|_\infty \leq r \}$$

S is closed $\rightarrow$ exercise



II need to show if $X \in S \Rightarrow PX \in S$

equivalently

if $\|X(t) - X_0\| \leqslant r$ for all $t \in [t_0, t_0+\delta]$ $\Rightarrow$ $\|PX(t) - X_0\| \leqslant r$ for all $t \in [t_0, t_0+\delta]$

$$PX(t) - X_0 = \int_{t_0}^{t} f(s, X(s))\, ds$$

$$= \int_{t_0}^{t} [f(s, X(s)) - f(s, X_0) + f(s, X_0)]\, ds$$

$$\Rightarrow \|PX(t) - X_0\| \leqslant \int_{t_0}^{t} \|f(s, X(s)) - f(s, X_0)\| + \|f(s, X_0)\|\, ds$$

① f is piecewise continuous in $t \Rightarrow$ it is bounded

$$\Rightarrow \max_{t \in [t_0, t_1]} \|f(t, X_0)\| = h < \infty$$

② $\|X(t) - X_0\| \leqslant r$ and f is locally Lip. $\Rightarrow$

$\exists L > 0$ s.t. $\|f(t, X(t)) - f(t, X_0)\| \leqslant L\|X(t) - X_0\| \leqslant Lr$

$$\Longrightarrow \quad \|PX(t) - X_0\| \leq \int_{t_0}^{t} (Lr + h)\, ds$$

$$\leq (Lr + h)\, \delta$$

- So, in order to ensure $PX \in S$, we need

$$(Lr + h)\,\delta \leq r$$

$$\Longrightarrow \quad \delta < \frac{r}{Lr + h}$$

III P is contraction

$$|(PX)(t) - (PY)(t)| = \left| \int_{t_0}^{t} f(s, X(s)) - f(s, Y(s))\, ds \right|$$

$$\leq \int_{t_0}^{t} |f(s, X(s)) - f(s, Y(s))|\, ds$$

$$\overset{\text{Lip.}}{\leq} \int_{t_0}^{t} L\, \underbrace{|X(s) - Y(s)|}_{\leq \|X - Y\|_\infty}\, ds$$

$$\leq L\, \|X - Y\|_\infty\, \delta$$

$$\Rightarrow \quad \| PX - PY \|_\infty \leq L\delta \| X - Y \|_\infty$$

- In order to have contraction $\rightarrow L\delta < 1$

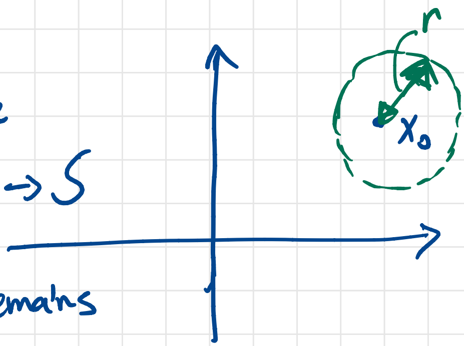
$$\Rightarrow \delta < \frac{1}{L}$$

- Combining the two conditions for δ

$$\delta < \min \left\{ \frac{1}{L}, \frac{r}{Lr + h} \right\}$$

- Then, CMT applies, $\exists!$ solution on S

or on the interval $[t_0, t_0 + \delta]$



(I) all trajectories inside the ball is closed set $\rightarrow S$

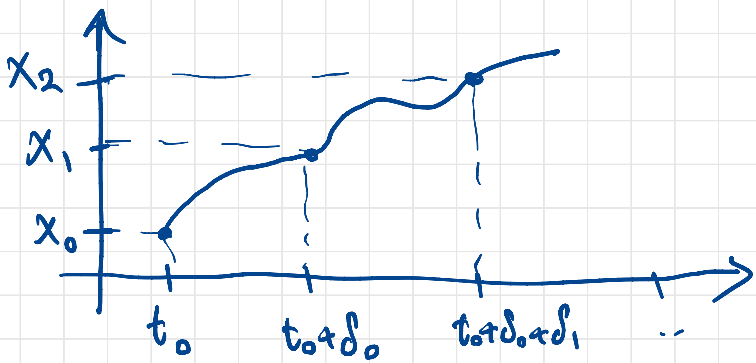
(II) Starting at X_0, $X(t)$ remains in ball if $\delta \leq \frac{r}{Lr+h}$, L is Lip. Const. for ball

(III) Contraction if $\delta \leq \frac{1}{L}$

proof of globall existence:

- why can't we conclude globall existence?

- we start at $x_0 \longrightarrow$ construct the solution for $[t_0, t_0+\delta_0]$
  then we take $x(t_0+\delta_0)$ as the new initial condition
  and construct solution for $[t_0+\delta_0, t_0+\delta_0+\delta_1]$
  and so on ...



- The issue is that $t_0+\delta_0+\delta_1+\delta_2+\delta_3+\cdots \not\Rightarrow \infty$
  might not extend to ∞

$$\text{e.g. } \delta_K = \frac{1}{2^K} \Rightarrow t_0 + \sum_K \delta_K \leq t_0 + 1$$

- but if f is globally Lip. $\exists$ universal Lip. constant L
- Two conditions:

$$\delta \leq \frac{1}{L}$$

$$\delta \leq \frac{r}{Lr+h} \xrightarrow{r \to \infty} \frac{1}{L}$$

$\Longrightarrow$ we can take $\delta_0 = \delta_1 = \delta_2 = \cdots = \frac{1}{L}$

$\Longrightarrow$ solution can be extended indefinitely.

<u>Remark</u>: if one knows (apriori) that the solution is bounded, then locally Lip. is enough to ensure global existence

- In other words, if no global solution, then there should be finite-time blow-up

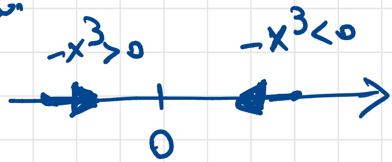
like $\dot{x} = x^2$

Example: $\dot{x} = -x^3$, $x(0) = x_0$

$\Rightarrow f(x) = x^3 \Rightarrow \underbrace{f'(x) = 3x^2}_{\text{locally Lip.}}$

$\Rightarrow$ local existence

But, we can argue that the solution is always bounded.



$\Rightarrow$ by remark, we have global existence

in fact, explicit form is known

$$X(t) = \operatorname{sgn}(X_0)\sqrt{\frac{X_0^2}{1+2(t-t_0)X_0^2}}$$

Example:

$$\dot{x}(t) = \underbrace{A(t)x(t) + g(t)}_{f(t,x)}$$

where $\underset{\text{matrix}}{A(t)}$, $\underset{\text{vetor}}{g(t)}$ are piecewise Cont. in t

$\Longrightarrow$ over any finite interval, $A(t)$ and $g(t)$ is bounded.

$\Rightarrow$ $\|A(t)\| \leq a$ where $\|\cdot\|$ is any induced matrix norm.

$$\|f(t,x) - f(t,y)\| = \|A(t)(x-y)\|$$

$$\leq \|A(t)\| \, \|x-y\|$$

$$\leq a\|x-y\| \quad \forall x, y \in \mathbb{R}^n$$

$\Rightarrow$ global Lip.

$\Rightarrow$ $\exists!$ solution on $[t_0, t_1]$

$\Rightarrow$ t_1 can be arbitrary large.

- Norm of matrices induced from vector norm.

- Consider norm $\|x\|_p = [x_1^p + x_2^p + \cdots + x_n^p]^{\frac{1}{p}}$ on $x \in \mathbb{R}^n$

- Consider a $n \times n$ matrix A.

- The norm of A induced from $\|\cdot\|_p$ is

$$\|Ax\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

- By definitin, we have the inequality

$$\|Ax\|_p \leqslant \|A\|_p \|x\|_p$$

- Special cases:

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |A_{ij}|$$ maximum absolute column sum

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |A_{ij}|$$ max abs. row sum

$$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)}$$ maximum singular value

- Important inequality

$$\|A\|_2 \leq \underbrace{\sqrt{tr(AA^T)}}_{\text{Frobinius norm}} = \sqrt{\sum_{ij} A_{ij}^2}$$